ON THE STABILITY OF A FREE RIGID BODY WITH A CAVITY FILLED WITH AN INCOMPRESSIBLE VISCOUS FLUID

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In the literature on questions connected with the motion of free mechanical systems in a central force field, for instance in [1], the problem of passive stabilization of such systems is formulated, as well as that of damping of small external disturbances, a viscous fluid being used as the damper.

Here we consider the problem of the behavior of such a system, the stability of its motion with regard to some known parameters, in the case when it possesses a cavity filled with a viscous incompressible fluid.

Sufficient stability conditions of circular motion of the mass center of the system and the conditions of equilibrium with a system of cavity filled with incompressible viscous fluid are obtained in the paper. The form and the method of solution of the stability of motion of rigid bodies with a liquid inclusion suggested by Rumiantsev [2] are used.

For a system which consists of a single rigid body, the sufficient conditions of stability were found by Beletskii [3].

1. We consider the problem of motion of a free, mechanical system which consists of a rigid body with a cavity, filled completely with a viscous, incompressible fluid, placed in a Newtonian central field of force.

Let O be the fixed attracting center with which we associate a fixed system of coordinates ξ, η, ζ.

The rigid body and the fluid which fills the cavity is considered as a single mechanical system whose kinetic energy is the sum of the kinetic energies of the rigid body and the fluid $T = T_1 + T_2$, while the moment of momentum is the geometrical sum of the moments of momentum of the rigid body and the fluid $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$. Subscript *1* always refers to quantities concerning the rigid body and the subscript *2* to quantities regarding the fluid.

Let G be the mass center of the system and R the radius vector from O to G.

The rigid body without the fluid has mass M_1, the mass of the fluid is M_2, its density is ρ and the coefficient of viscosity μ^*. The mass of the system is equal $M = M_1 + M_2$.

Let the origin of the moving system of coordinates x, y, z be in G, and let the direction of the axes be along the principal axes of the central ellipsoid of inertia of the system considered, the axes of the latter being the principal axes of inertia for both the rigid body and the fluid. Thus, if A, B, C are the principal moments of inertia of the system, then

$$A = A_1 + A_2, \qquad B = B_1 + B_2, \qquad C = C_1 + C_2$$

Let the projection of the instantaneous angular velocity of the system with respect to the center of mass on the moving axes be p, q, r.

Let us note further, that, as is usual, the orientation of the system of coordinate axes x, y, z with respect to axes ξ, η, ζ, is given by a system of direction cosines α_i, β_i, γ_i $(i = 1, 2, 3)$, and with respect to the radius vector R by the system of direction cosines β, β', β''.

The moment of momentum with respect to the fixed center is equal to

$$\mathbf{K}_0 = \mathbf{R} \times M\mathbf{V} + \mathbf{K} \qquad (R^2 = \xi^2 + \eta^2 + \zeta^2, V = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)$$

where $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$ is the moment of momentum of the system evaluated with respect to G, in its motion with respect to the Koenig system of coordinates. The projections on the moving axes are

$$K_{1x} = A_1 p, \qquad K_{1y} = B_1 q, \qquad K_{1z} = C_1 r$$

If u, v, w designate the projection of the moving axes of the relative velocity of the fluid as it moves with respect to the rigid body, then

$$K_{2x} = \rho \int_\tau (yV_z - zV_y)\, d\tau, \quad K_{2y} = \rho \int_\tau (zV_x - xV_z)\, d\tau, \quad K_{2z} = \rho \int_\tau (xV_y - yV_x) d\tau$$

where

$$V_x = u + qz - ry, \quad V_y = v + rx - pz, \quad V_z = w + py - qx$$

and the integration is carried out over the total volume of the fluid τ.

Introducing the notations g_x, g_y, g_z for the projection on the moving axes x, y, z of the angular momentum vector of the relative motion of the fluid (with respect to the shell) we may finally write

$$K_{2x} = A_2 p + g_x, \qquad K_{2y} = B_2 q + g_y, \qquad K_{2z} = C_2 r + g_z$$

$$g_x = \rho \int_\tau (yw - zv)\, d\tau, \quad g_y = \rho \int_\tau (zu - xw)\, d\tau, \quad g_z = \rho \int_\tau (xv - yu)\, d\tau$$

The force function of the forces acting on the system, is determined by the integral (μ is the gravitational constant)

$$U = \int_M \frac{\mu dm}{\rho_m} \qquad ({\rho_m}^2 = R^2 + 2R(x\beta + y\beta' + z\beta'') + x^2 + y^2 + z^2)$$

Let us consider the problem in a limited form, in the sense that instead of considering the indicated force function U we will consider an approximate expression obtained by expanding U in a series of powers x/R, y/R, z/R, neglecting all terms higher than of the second order, which is justified by the fact that for real systems their characteristic dimension l is much smaller than R, namely of the order $l/R \approx 10^{-4} \div 10^{-6}$.

Thus, we have [3]

$$U = \frac{\mu M}{R} - \frac{3}{2}\frac{\mu}{R^3}(A\beta^2 + A\beta'^2 + C\beta''^2) + \frac{3}{2}\frac{\mu}{R^3}\frac{A + B + C}{3} \tag{1.1}$$

Now we can write the equations of motion of the system considered

$$M\ddot{\xi} = \frac{\partial U}{\partial \xi}, \qquad M\ddot{\eta} = \frac{\partial U}{\partial \eta}, \qquad M\ddot{\zeta} = \frac{\partial U}{\partial \zeta} \tag{1.2}$$

$$A\frac{dp}{dt} + (C - B)qr + \frac{dg_x}{dt} + qg_z - rg_y = M_x \quad (xyz,\ ABC,\ pqr) \tag{1.3}$$

$$\frac{d}{dt}(V_x + u + qz - ry) + q(V_y + w + py - qx) - r(V_y + v + rx - pz) =$$
$$= F_x - \frac{1}{\rho}\frac{\partial P^*}{\partial x} + \nu \Delta u \qquad (xyz,\ uvw,\ pqr) \qquad \left(\nu = \frac{\mu^*}{\rho}\right) \tag{1.4}$$

Here the symbols $(xyz,\ ABC,\ pqr)$ and $(xyz,\ uvw,\ pqr)$ indicate that two other equations are obtained by simultaneous circular permutation of the indicated letters, and dots above the variables indicate differentiation with respect to time. Equations (1.2) to (1.4) must be supplemented by the equation of incompressibility

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1.5}$$

and the conditions on the walls S of the cavity

$$u=v=w=0 \tag{1.6}$$

The equations of motion are completed by the well known kinematic equations of Poisson for the direction cosines. The notations in (1.2) and (1.3) are the usual ones: P is the hydrodynamic pressure, v_x, v_y, v_z are the projections of the velocity of G on the moving axes; F_x, F_y, F_z are the projections on the moving axes of the Newtonian force, and M_x, M_y, M_z are the projections of the gravitational moments which in the approximation selected are of the form

$$M_x=3\,\frac{\mu}{R^3}(C-B)\,\beta''\beta' \qquad (ABC,\ \beta\ \beta'\ \beta'')$$

We note that the equations of motion of the problem considered admit an integral $\mathbf{K}_0$ = const and several trivial integrals among which we indicate only two

$$\beta^2+\beta'^2+\beta''^2=1, \qquad \gamma_1^2+\gamma_2^2+\gamma_3^2=1 \tag{1.7}$$

where γ_1, γ_2, γ_3 are the direction cosines of the ζ-axis with respect to the axes of the moving system.

Selecting the ζ-axis as being orthogonal to the orbital plane, we may write the area integral in the following form:

$$M(\xi\dot\eta-\eta\dot\xi)+(Ap+g_x)\,\gamma_1+(Bq+g_y)\,\gamma_2+(Cr+q_z)\,\gamma_3=\text{const} \tag{1.8}$$

2. Let us formulate several relations. Multiplying Equations (1.2) by $\dot\xi$, $\dot\eta$, $\mathring{\zeta}$ respectively, Equations (1.3) by p, q, r and adding the results, and multiplying (1.4) by u, v, w; then adding and integrating the last expression obtained over the whole fluid volume, adding all the results and taking into consideration conditions (1.5) and (1.6) we obtain

$$\frac{d}{dt}(T-U)=-\mu^*\int_\tau\left[\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2\right]d\tau \tag{2.1}$$

Here

$$2T = M(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) + A_1p^2 + B_1q^2 + C_1r^2 + A_2p^2 + B_2q^2 + C_2r^2 +$$
$$+ \rho\int_\tau (u^2 + v^2 + w^2)\, d\tau + 2\rho\int_\tau [u(qz - ry) + v(rx - pz) + w(py - qx)]\, d\tau$$

It follows from this

$$T - U \leqslant T_0 - U_0 \qquad (T_0 = T|_{t=0} U_0 = U|_{t=0})$$

The quality sign will be valid if the system moves as one single rigid body, or if the fluid is an ideal one ($\mu^* = 0$). Using transformations [2]

$$\omega_1 = K_{2x} / A_2, \qquad \omega_2 = K_{2y} / B_2, \qquad \omega_3 = K_{2z} / C_2 \tag{2.3}$$

The new unknown functions $\omega_i(t)$ are determined if V_x, V_y, V_z are known. Let us introduce further unknowns

$$V_1 = V_x + \omega_3 y - \omega_2 z, \quad V_2 = V_y + \omega_1 z - \omega_3 x, \quad V_3 = V_z + \omega_2 x - \omega_1 y \tag{2.4}$$

By definition of $\omega_i(t)$

$$\rho\int_\tau (yV_3 - zV_2)\, d\tau = \rho\int_\tau (zV_1 - xV_3)\, d\tau = \rho\int_\tau (xV_2 - yV_1)\, d\tau = 0$$

Obviously, V_x, V_y, V_z will be determined if $\omega_i(t)$ and $V_i(t, x, y, z)$ ($i = 1, 2, 3$) are known.

Now a part of the kinetic energy of the system, associated with the motion of the fluid with respect to the mass center of the whole system, is reduced to the convenient form

$$2T_2 = \frac{K_{2x}^2}{A_2} + \frac{K_{2y}^2}{B_2} + \frac{K_{2z}^2}{C_2} + \rho\int_\tau (V_1^2 + V_2^2 + V_3^2)\, d\tau \tag{2.5}$$

If the spherical coordinates of the mass center are introduced

$$\xi = R\cos\psi\cos\varphi, \qquad \eta = R\cos\psi\sin\varphi, \qquad \zeta = R\sin\psi$$

then the first integrals of the equation of motion of the system may be written finally in the form

$$M(\dot{R}^2 + R^2\dot{\psi}^2 + R^2\cos^2\psi\dot{\varphi}^2) + A_1p^2 + B_1q^2 + C_1r^2 +$$
$$+ \frac{K_{2x}^2}{A_2} + \frac{K_{2y}^2}{B_2} + \frac{K_{2z}^2}{C_2} + \rho\int_\tau (V_1^2 + V_2^2 + V_3^2)\, d\tau - 2U \leqslant \mathrm{oonst}$$

$$MR^2 \cos^2 \psi\dot{\varphi}+(A_1p+K_{2x})\,\gamma_1 + (B_1q + K_{2y})\,\gamma_2 + (C_1r + K_{2z})\,\gamma_3 = \text{const}$$

$$\beta^2 + \beta'^2 + \beta''^2 = 1, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

3. The equations of motion of the system considered admit the particular solution

$$\begin{gathered} p = q = 0, \qquad r = \omega, \qquad u = v = w = 0 \\ \beta = \beta'' = 0, \qquad \beta' = 1, \qquad K_{2x} = K_{2y} = 0, \qquad K_{2z} = C_2\omega \\ \gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1, \quad R = R_0, \quad \dot{R} = 0, \quad \psi = 0, \quad \dot{\psi} = 0, \quad \dot{\varphi} = \omega \end{gathered} \tag{3.1}$$

This particular solution corresponds to the motion of the system on a circular orbit $R = R_0$ with constant angular velocity ω, such that the principal central axes of the system are along the tangent, the radius vector and the binormal of the undisturbed orbit. The fluid is hereby at rest with the body, i.e. the system moves as a single rigid body.

Let us investigate the stability of the undisturbed motion of the system with respect to the variables

$$p, \ q, \ r; \quad \beta, \ \beta', \ \beta''; \quad \gamma_1, \ \gamma_2, \ \gamma_3; \quad K_{2x}, \ K_{2y}, \ K_{2z}; \quad R, \ \dot{R}, \ \psi, \ \dot{\psi}, \ \dot{\varphi} \tag{3.2}$$

Using the same notations for quantities which in the undisturbed motion have trivial values, we set for the disturbed motion

$$\begin{gathered} r = \omega + x_1, \qquad \beta' = 1 + x_3, \qquad \gamma_3 = 1 + x_2, \qquad K_{2z} = C_2\omega + x_4 \\ R = R_0 + x_5, \qquad \dot{R} = \dot{x}_5, \qquad \dot{\varphi} = \omega + \dot{x}_6 \end{gathered}$$

In the undisturbed motion we had also the formula

$$\omega_1 = \omega_2 = 0, \qquad \omega_3 = \omega, \qquad V_1 = V_2 = V_3 = 0$$

The equations of perturbed motion of the problem considered admit the first integrals

$$\begin{aligned} W_1 = {} & M\dot{x}_5^2 + MR_0^2\dot{\psi}^2 + MR_0^2\dot{x}_6^2 + \Big(M\omega^2 - \frac{2\mu M}{R_0^3} + \frac{18\mu B}{R_0^5} - \\ & - \frac{6\mu\,(A+B+C)}{R_0^5}\Big)\,x_5^2 - MR_0^2\omega^2\psi^2 + A_1p^2 + B_1q^2 + C_1x_1^2 + \\ & + \frac{K_{2x}^2}{A_2} + \frac{K_{2y}^2}{B_2} + \frac{x_4^2}{C_2} + \frac{3\mu A}{R_0^3}\,\beta^2 + \frac{3\mu C}{R_0^3}\,\beta''^2 + \\ & + \frac{3\mu B}{R_0^3}\,x_3^2 + 4MR_0\,\omega x_5\dot{x}_6 - \frac{18\mu B}{R_0^4}\,x_3x_5 + \rho\int_\tau (V_1^2 + V_2^2 + V_3^2)\,d\tau + \end{aligned}$$

$$+2MR_0^2\omega\dot{x}_6+\left(2MR_0\omega^2+\frac{2\mu M}{R_0^2}-\frac{9\mu B}{R_0^4}+\frac{3\mu(A+B+C)}{R_0^4}\right)x_5+$$
$$+2C_1\omega x_1+2\omega x_4+\frac{6\mu B}{R_0^3}x_3+O\,(3)\leqslant \text{const} \tag{3.3}$$

$$W_2=M\omega x_5^2-MR_0^2\omega\psi^2+2MR_0x_5\dot{x}_6+A_1p\gamma_1+K_{2x}\gamma_1+$$
$$+B_1q\gamma_2+K_{2y}\gamma_2+x_4x_2+C_1x_1x_2+MR_0^2\dot{x}_6+$$
$$+2MR_0\omega x_5+C_1x_1+x_4+C\omega x_2+O\,(3)=\text{const} \tag{3.4}$$

$$W_3=\beta^2+\beta''^2+x_3^2+2x_3=\text{const} \tag{3.5}$$

$$W_4=\gamma_1^2+\gamma_2^2+x_2^2+2x_2=\text{const} \tag{3.6}$$

By $O\ (3)$ we indicate terms not lower than of third order of magnitude with respect to the perturbation.

For the disturbed motion $dW_1/dt \leqslant 0$ is valid (2.1).

Let us consider a function of the variables of the problem, constructed by the method of Chetaev [4] in the form of a relation of the first integrals of the equation of motion

$$W=W_1-2\omega W_2-\frac{3\mu B}{R_0^3}W_3+C\omega^2W_4+\sigma W_2^2+\lambda W_3^2+\delta W_4^2= \tag{3.7}$$
$$=M\dot{x}_5^2+MR_0^2\dot{\psi}^2+MR_0^2\omega^2\psi^2+\frac{3\mu}{R_0^3}(A-B)\beta^2+\frac{3\mu}{R_0^3}(C-B)\beta''^2+$$
$$+A_1p^2-2\omega A_1p\gamma_1+C\omega^2\gamma_1^2-2\omega\gamma_1K_{2x}+\frac{1}{A_2}K_{2x}^2+B_1q^2-$$
$$-2\omega B_1q\gamma_2+C\omega^2\gamma_2^2-2\omega\gamma_2K_{2y}+\frac{1}{B_2}K_{2y}^2+4\lambda x_3^2+$$
$$+\left(\frac{2\mu M}{R_0^3}-5\omega^2M+4\sigma M^2R_0^2\omega^2\right)x_5^2+(MR_0^2+\sigma M^2R_0^4)\,\dot{x}_6^2+$$
$$+(C\omega^2+\sigma C^2\omega^2+4\delta)\,x_2^2+(C_1+\sigma C_1^2)\,x_1^2+\left(\frac{1}{C_2}+\sigma\right)x_4^2-$$
$$-\frac{18\mu B}{R_0^4}x_5x_3+4\sigma M^2R_0^3\omega x_5\dot{x}_6+4\sigma MR_0C\omega^2x_5x_2+$$
$$+4\sigma MR_0C_1\omega x_5x_1+4\sigma MR_0\omega x_5x_4+2\sigma MR_0^2C\omega x_2\dot{x}_6+$$
$$+2\sigma MR_0^2C_1x_1\dot{x}_6+2\sigma MR_0^2x_4\dot{x}_6+(-2\omega C_1+2\sigma CC_1\omega)\,x_1x_2+$$
$$+(-2\omega+2\sigma C\omega)\,x_2x_4+2\sigma C_1x_1x_4+\rho\int_\tau(V_1^2+V_2^2+V_3^2)\,d\tau+O\,(3)$$

Here σ, λ and δ are constant quantities. The angular velocity of ~~the~~ motion of the mass center of the system is determined by the relation

$$\omega^2 = \frac{\mu}{R^3} - \frac{9\mu B}{2MR_0^5} + \frac{3}{2}\,\frac{\mu(A+B+C)}{MR_0^5} \qquad (3.8)$$

By Sylvester's criterion, for the function W to be positive-definite the following conditions are necessary and sufficient:

$$A > B,\quad C > B;\quad C > A_1,\quad C > A_1 + A_2;\quad C > B_1,\quad C > B_1 + B_2 \qquad (3.9)$$

and further, the diagonal minors must be positive for

$$\|h_{ij}\| \qquad (h_{ij} = h_{ji}) \qquad (i, j = 1, \ldots, 6) \qquad (3.10)$$

$$h_{11} = \frac{2\mu M}{R_0^3} - 5\omega^2 M + 4\sigma M^2 R_0^2 \omega^2, \qquad h_{12} = -\frac{9\mu B}{R_0^4}, \qquad h_{13} = 2\sigma M^2 R_0^3 \omega$$

$$h_{14} = 2\sigma M R_0 C\omega^2, \qquad h_{15} = 2\sigma M R_0 C_1 \omega, \qquad h_{16} = 2\sigma M R_0 \omega$$

$$h_{22} = 4\lambda, \qquad h_{23} = h_{24} = h_{25} = h_{26} = 0$$

$$h_{33} = MR_0^2 + \sigma M^2 R_0^4, \qquad h_{34} = \sigma M R_0^2 C\omega, \qquad h_{35} = \sigma M R_0^2 C_1$$

$$h_{36} = \sigma M R_0^2, \qquad h_{44} = C\omega^2 + \sigma C^2\omega^2 + 4\delta, \qquad h_{45} = -\omega C_1 + \sigma C C_1 \omega$$

$$h_{46} = -\omega + \sigma C\omega, \qquad h_{55} = C_1 + \sigma C_1^2, \qquad h_{56} = \sigma C_1, \qquad h_{66} = \frac{1}{C_2} + \delta$$

A suitable selection of the constants σ, λ and δ may insure the positiveness of these minors, whereby the inequalities obtained are not essential and limit only the selection of the constants indicated. The inequalities (3.9), however, give jointly

$$C > A > B \qquad (3.11)$$

Thus, if $C > A > B$, then σ, λ and δ may be selected such that the quadratic integral W will be positive-definite with respect to the variables

$$p_1 q,\ x_1,\ \beta,\ \beta'',\ x_3,\ \gamma_1,\ \gamma_2,\ x_2,\ K_{2x},\ K_{2y},\ K_{2z},\ x_4,\ x_5,\ \dot{x}_5,\ \psi,\ \dot{\psi},\ \dot{x}_6$$

This will be Liapunov's function for the problem. Indeed, dW/dt, taken in place of the equation of disturbed motion, will not be positive as follows from Equation (2.1). And this, in accordance with Liapunov's stability theorem, allows us to estimate the stability of the indicated undisturbed motion (3.1) of the rigid body with a cavity, filled with a viscous, incompressible fluid.

The sufficient stability conditions obtained coincide in their form with the sufficient conditions of the stability of the system consisting of a single rigid body [3], but in the case considered

$$C_1 + C_2 > A_1 + A_2 > B_1 + B_2$$

If for a single rigid body, which which, for instance, $A_1 > C_1 > B_1$, condition of the type (3.1) is not satisfied, then it is easily seen that for a rigid body with a cavity, filled with a fluid, selecting the cavity, such that $C_2 > A_2 \geqslant B_2$, the satisfaction of conditions (3.10) may be achieved, conserving the mass geometry of the rigid body itself, which might be necessary for other reasons.

We note that the sufficient stability conditions found do not include terms related to the viscosity of the fluid; they merely limit the selection of the mass geometry of the system.

The results obtained not only give the sufficient stability conditions, but allow broader conclusions to be drawn, which are based on the theorem of Zhukovskii [5].

Zhukovskii has shown that in the presence of a relative motion of the fluid, the energy of the system is dissipated (this follows from (2.1)), and, therefore, two possiblities can occur: either the energy of the system will always decrease and the system will finally come to rest, or the system will approach a pure rotation with constant angular velocity around one of its principal axes of inertia as one single rigid body. The first possibility is excluded because the area integral (1.13) exists. Consequently, if the vector $\mathbf{K}_0$ is unperturbed, then any perturbing motion corresponding to this condition will asymptotically approach that consideration in the problem. If, however, the vector $\mathbf{K}$ is disturbed, then the disturbing motion approaches asymptotically a certain new steady motion corresponding to the changed moment of momentum.

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